UDC 534

ON A MODIFICATION OF THE AVERAGING METHOD AND ESTIMATES OF HIGHER APPROXIMATIONS

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The asymptotic method of multiscale expansions (see [1, 2] for ordinary differential equations is expounded and substantiated. It is shown that the methods of multiscale expansions and of averaging [3] yield equivalent results in any approximation. The findings about convergence in finite time intervals obtained in [4, 5] are generalized. It is shown that the time interval in which the error of an expansion remains small substantially depends on the properties and stability of the approximate solution.

The methods of Bogoliubov and Mitropolskii are well substantiated in [3 -5] and the order of closeness between the exact solution and its first [3] and higher [4, 5] approximations is established. Construction of higher approximations is, as a rule, very laborious. The method of multiscale expansions based on qualitative concepts of motion properties of systems makes it, on the other hand, possible to obtain a higher approximation without having to resort to cumbersome calculations. Furthermore, it gives a clearer picture of the physical essence of motion by separating "quick-acting" and "slow" effects that occur in various intervals of time. It was proved on specific examples that solutions derived by the method of multiscale expansions and those obtained by the method of averaging coincide in every approximation, but this has not been proved for the general case and any number of approximations. Below we show that solutions obtained by both these methods are completely equivalent, and that the theorems on the existence and convergence of asymptotic expansions that are valid in the method of averaging are, also, applicable in the method of multiscale expansions.

1. Let E be an n-dimensional real space and D a bounded region in it. We consider the equation in its standard form

$$dy / dt = \varepsilon Y_0 (t, y) + \varepsilon^2 Y_1 (t, y) + \ldots + \varepsilon^k Y_{k-1} (t, y) + \varepsilon^{k+1} Y_k (t, y, \varepsilon) \quad 0 \leqslant t \leqslant T, \quad y \in D \tag{1.1}$$

Operators Y_i $(i = 0, \ldots, k)$ are continuous with respect to y and have $k - i$ derivatives in D with respect to t and ε which are measurable.

We propose to seek for that equation a solution of the form

$$y = f_0 (t, \tau_1, \tau_2, \ldots) + \varepsilon F_1 (t, \tau_1, \tau_2, \ldots) + \ldots + \varepsilon^k F_k (t, \tau_1, \tau_2, \ldots) \tag{1.2}$$

where $\tau_1 = \varepsilon t, \ldots, \tau_m = \varepsilon^m t$ are slow variables which define motions that take place at various velocities. The different rate of change of variables is taken into account in the differentation

$$\frac{dy}{dt} = \frac{\partial f_0}{\partial t} + \varepsilon \left(\frac{\partial f_0}{\partial \tau_1} + \frac{\partial F_1}{\partial t}\right) + \varepsilon^2 \left(\frac{\partial f_0}{\partial \tau_2} + \frac{\partial F_1}{\partial \tau_1} + \frac{\partial F_2}{\partial t}\right) + \cdots \tag{1.3}$$

For an unambiguous determination of expansion coefficients they are subjected to the condition of expansion homogeneity

$$\frac{\|F_1\|}{\|f_0\|} \sim 1, \ldots, \frac{\|F_i\|}{\|F_{i-1}\|} \sim 1 \quad \text{for} \quad 0 \leqslant t < \infty, \quad \varepsilon \to 0 \quad (i = 1, \ldots, \qquad (1.4)$$

The fulfilment of that condition ensures the closeness of solution to the generating solution in the related time interval [2].

The substitution of (1.2) and (1.3) into (1.1) and the equating of coefficients at like powers of ε yields a system for the successive determination of operators $f_0, F_1, \ldots, F_k$.

The zero approximation equation $\partial f_0 / \partial t = 0$ implies that f_0 does not explicitly depend on t and is a function of slow variables $f_0 = f_0(\tau_1, \tau_2, \ldots)$. This function is so far unknown and is to be determined by higher approximation equations with allowance for condition (1.4). For $F_1, \ldots F_k$ we have

$$\frac{\partial F_1}{\partial t} = Y_0(t, f_0) - \frac{\partial f_0}{\partial \tau_1} \qquad (1.5)$$

$$\frac{\partial F_2}{\partial t} = Y_1(t, f_0) + \frac{\partial Y_0}{\partial f_0} F_1 - \frac{\partial F_1}{\partial \tau_1} - \frac{\partial f_0}{\partial \tau_2} = \Phi_1(t, f_0) - \frac{\partial f_0}{\partial \tau_2}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial F_k}{\partial t} = \Phi_{k-1}(t, f_0) - \frac{\partial f_0}{\partial \tau_k}$$

$$\Phi_m(t, f_0) = \sum_{j=0}^{m-i} P_{ij} - \sum_{r=1}^{m} \frac{\partial F_{m-r+1}}{\partial \tau_r} \qquad (1.6)$$

Operators P_{ij} are determined by the expansions

$$Y_i(t, f_0 + \varepsilon F_1 + \ldots + \varepsilon^k F_k) = \sum_{j=0}^{k-i} P_{ij}\varepsilon^j + P_i(\varepsilon)\,\varepsilon^{k-i+1} \qquad (1.7)$$

It follows from the first equation of system (1.5) that for condition (1.4) to be satisfied it is necessary to eliminate from the equality

$$F_1 = \int_0^t Y_0(s, f_0)\, ds - t\frac{\partial f_0}{\partial \tau_1}$$

terms that are linear to t by setting

$$\frac{\partial f_0}{\partial \tau_1} = \overline{\Phi}_0(f_0), \quad \overline{\Phi}_0(f_0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T Y_0(f_0, s)\, ds = \overline{Y}_0(f_0) \qquad (1.8)$$

Thus

$$F_1(t, \tau_1, \ldots) = \int_0^t [Y_0(f_0, s) - \overline{\Phi}_0(f_0)]\, ds = F_1(t, f_0(\tau_1, \ldots)) \qquad (1.9)$$

By excluding from all subsequent approximations all secular terms we obtain

$$\frac{\partial f_0}{\partial \tau_1} = \overline{\Phi}_0(f_0), \quad \overline{\Phi}_0(f_0) = \lim_{T\to\infty} \frac{1}{T}\int_0^T Y_0(f_0, s)\, ds = \overline{Y}_0(f_0) \qquad (1.10)$$

$$F_i(t, \tau_1, \ldots) = F_i(t, f_0(\tau_1, \ldots)) = \int_0^t [\Phi_{i-1}(s, f_0) - \overline{\Phi}_{i-1}(f_0)]\, ds \qquad (1.11)$$

Taking into account that $\tau_1, \tau_2, \ldots, \tau_k$ are dependent variables, we write the equations for determining f_0 as

$$\frac{df_0}{dt} = \varepsilon \frac{\partial f_0}{\partial \tau_1} + \ldots + \varepsilon^k \frac{\partial f_0}{\partial \tau_k} = \varepsilon\overline{\Phi}_0(f_0) + \ldots + \varepsilon^{k+1}\overline{\Phi}_k(f_0) \tag{1.12}$$

Higher terms of expansion are determined by equalities (1.11) in which function f_0 is already known from the solution of Eq. (1.12).

Equation (1.12) and formulas (1.11) determine f_0 and F_i to within terms of order ε^k. The explicit introduction of slow variables discloses the physical essence of solution, namely, that Eqs. (1.10) define slow processes that are significant only in the time intervals $t \sim T / \varepsilon^{i+1}$, and that by retaining in expansion (1.2) $k+1$ terms, we take into account not only the minor but, also, the slow effects that appear in the time intervals $t \sim T / \varepsilon^k$.

If f_0 is taken as the new variable which defines operators F_i by formulas (1.11), then, by substituting (1.2) and (1.3) into Eq. (1.1) and taking into account formulas (1.6) and (1.12) we find that f_0 satisfies the exact equation

$$df_0 / dt = \varepsilon\overline{\Phi}_0(f_0) + \ldots + \varepsilon^k\overline{\Phi}_{k-1}(f_0) + \varepsilon^{k+1}\Phi_k(t, f_0) + \varepsilon^{k+1}R(t, f_0, \varepsilon) \tag{1.13}$$

where $\lim R(t, f_0, \varepsilon) = 0$ when $\varepsilon \to 0$. If, however, f_0 is determined by the averaged equation (1.12), we are faced with the error of the expansion whose estimate is given below in Sect. 2.

Let us show that the principal term of expansion of f_0 and the higher approximations expressed as functions of it are exactly the same as the coefficients of asymptotic expansion obtained by the method of averaging.

In the method of averaging the solution of Eq. (1.1) is sought in the form of expansion [3-5]

$$y = x + \varepsilon U_1(t, x) + \ldots + \varepsilon^k U_k(t, x) \tag{1.14}$$

where the principal term of expansion x is taken as the new variable. Following the basic assumptions and reasoning in [5] and substituting (1.14) we pass from (1.1) to the autonomous equation

$$dx / dt = \varepsilon X_0(x) + \varepsilon^2 X_1(x) + \ldots + \varepsilon^k X_{k-1}(x) + \varepsilon^{k+1} X_k(t, x, \varepsilon) \tag{1.15}$$

accurate to within terms of order ε^{k+1}.

Operators $X_i(x)$, and $U_{i+1}(t, x)$ are successively determined by formulas [4,5]

$$X_i(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \Psi_i(s, x)\, ds \tag{1.16}$$

$$U_{i+1}(t, x) = \int_0^t [\Psi_i(s, x) - X_i(x)]\, ds$$

where

$$\Psi_\alpha(t, x) = \sum_{i+j+l=\alpha} Q_l P_{ij} - \sum_{0\leqslant m\leqslant \alpha-1} Q_{\alpha-m}\frac{\partial U_{m+1}}{\partial t} \quad (\alpha = 0, \ldots, k) \tag{1.17}$$

$$Q_l = -\sum_{s+p=l} \frac{\partial U_s}{\partial x} Q_p \left(\left[I + \varepsilon\frac{\partial U_1}{\partial x} + \ldots + \varepsilon^k \frac{\partial U_k}{\partial x}\right]^{-1} = \sum_{i=0}^{\infty} \varepsilon^i Q_i\right) \tag{1.18}$$

Operators P_{ij} satisfy formulas (1.7). Operator $X_k(t, x, \varepsilon)$ is determined by formula

$$X_k(t, x, \varepsilon) = \Psi_k(t, x) + L(t, x, \varepsilon), \quad \lim_{\varepsilon\to 0} L(t, x, \varepsilon) = 0$$

where $L(t, x, \varepsilon)$ is the remainder formed by the substitution into the input equation (1.1) of expansion (1.14) in which $x(t)$ is determined by the averaged equation

$$dx/dt = \varepsilon X_0(x) + \ldots + \varepsilon^k X_{k-1}(x) + \varepsilon^{k+1}\overline{X}_k(x) \tag{1.19}$$

$$\left(\overline{X}_k(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_k(t, x, 0)\, dt\right)$$

and operators X_i and U_i are specified by formulas (1.16). The formula for operator L is given in [3-5].

Comparison of formulas (1.10) - (1.13) with (1.14) - (1.18) shows that solutions derived by either method are the same, if the quantities Φ_i and Ψ_i determined, respectively, by formulas (1.6) and (1.17) are the same.

In fact, if $x^{(k+1)}$ and $f_0^{(k+1)}$ are, respectively, the solutions of Eqs. (1.19) and (1.12) and $\Phi_i = \Psi_i$, with $0 \leqslant i \leqslant m-1$, then

$$\frac{df_0^{(m)}}{dt} = \sum_{i=1}^m \varepsilon^i \Psi_{i-1}(f_0^{(m)}) = \sum_{i=1}^m \varepsilon^i \Phi_{i-1}(f_0^{(m)})$$

i.e. $f_0^{(m)} = x_0^{(m)}$, and, consequently,

$$F_i = U_i, \quad 1 \leqslant i \leqslant m \tag{1.20}$$

We apply the method of mathematical induction for $i = 0$ when $\Phi_0 = \Psi_0 = Y_0$. We shall prove that when

$$\Phi_{m-1} = \Psi_{m-1} = \sum_{i+j+l=m-1} Q_l P_{ij} - \sum_{0\leqslant r\leqslant m-2} Q_{m-r-1}\frac{\partial F_{r+1}}{\partial t} \tag{1.21}$$

where in accordance with (1.18) and (1.20)

$$Q_l = -\sum_{s+p=l} \frac{\partial F_s}{\partial f_0} Q_p, \quad 0 \leqslant l \leqslant m \tag{1.22}$$

then

$$\Phi_m = \Psi_m = \sum_{i+j+l=m} Q_l P_{ij} - \sum_{0\leqslant r\leqslant m-1} Q_{m-r}\frac{\partial F_{r+1}}{\partial t} \tag{1.23}$$

Let us, first, calculate the second sum in (1.6).

By virtue of (1.5) and (1.21)

$$\sum_{r=1}^m \frac{\partial F_{m-r+1}}{\partial \tau_r} = \sum_{r=1}^m \frac{\partial F_{m-r+1}}{\partial f_0}\frac{\partial f_0}{\partial \tau_r} = \sum_{r=1}^m \frac{\partial F_{m-r+1}}{\partial f_0}\left[\Phi_{r-1} - \frac{\partial F_r}{\partial t}\right] =$$

$$\sum_{r=1}^m \frac{\partial F_{m-r+1}}{\partial f_0}\left[\sum_{i+j+l=r-1} Q_l P_{ij} - \sum_{0\leqslant q\leqslant r-2} Q_{r-1-q}\frac{\partial F_{q+1}}{\partial t} - \frac{\partial F_r}{\partial t}\right]$$

and, with allowance for (1.22)

$$\sum_{r=1}^{m}\sum_{i+j+l=r-1}\frac{\partial F_{m-r+1}}{\partial f_0}Q_l P_{ij} = -\sum_{\substack{s+i+j=m\\ s\geqslant 1}} Q_s P_{ij} \tag{1.24}$$

Because $Q_0 = 1$ we can write

$$\sum_{j=0}^{m-i} P_{ij} + \sum_{\substack{s+i+j=m\\ s\geqslant 1}} Q_s P_{ij} = \sum_{\substack{s+i+j\\ s>0}} Q_s P_{ij} \tag{1.25}$$

and exactly in the same way

$$\sum_{r=1}^{m}\sum_{0\leqslant q\leqslant r-2}\frac{\partial F_{m-r+1}}{\partial f_0}\left[Q_{r-1-q}\frac{\partial F_{q+1}}{\partial t}+\frac{\partial F_r}{\partial t}\right] = \sum_{0\leqslant q\leqslant m-1} Q_{m-q}\frac{\partial F_{q+1}}{\partial t} \tag{1.26}$$

The substitution of (1.24)- (1.26) into (1.6) shows that (1.6) coincides with (1.23).

This proves that expansions (1.2) and (1.14) are equivalent, i.e. $f_0 = x$ and

$$F_i\,(t, \tau_1, \ldots) = F_i\,(t, f_0\,(\tau_1, \ldots)) = U_i\,(t, x).$$

The conditions under which operators U_i or (what is the same) F_i, can be successively determined were defined in [3-5]. In particular, if all operators $Y_i\ (i = 0, \ldots, k)$ together with derivatives of up to $k - i$ order are bounded in D if operators U_i are bounded, and if there exists averaging of operator Ψ_k, condition (P_k)), then operators U_i can be successively determined by formulas (1.16).

2. It follows from the analysis in Sect. 1 that, when determining the $k + 1$ terms of expansion, we retain the quantities that are important in the time interval $t \sim T / \varepsilon^k$.

Simultaneously the method of averaging and that of multiscale expansions conform to the theorem [3-5] which states that with specific constraints on coefficients of the equation for any (finite) T the following inequality is satisfied.

$$\lim_{\varepsilon\to 0}\ \sup_{x\in M(\varepsilon,\, T/\varepsilon)}\ \max_{0\leqslant t\leqslant T/\varepsilon}\ (\varepsilon^{-k}\,\|x\,(t) - \bar{x}\,(t)\|) = 0 \tag{2.1}$$

$$\bar{x} \in M_k\left(\varepsilon, \frac{T}{\varepsilon}\right)$$

where $M\ (\varepsilon, T)$ is the set of all solutions of Eq. (1.15) determinate in $[0, T]$ and $M_k\ (\varepsilon, T)$ is the set of all asymptotic approximation of the $k + 1$ order to solution $x\ (t)$. Of interest is the behavior of solution in the time interval $t \sim T / \varepsilon^k$, since it makes sense to retain only those terms that are important within the convergence range.

Let us again consider Eq. (1.1) whose solution is sought in the form (1.14). The principal term of expansion x satisfies the exact equation

$$dx / dt = \varepsilon X_0\,(x) + \varepsilon^2 X_1\,(x) + \ldots + \varepsilon^{k+1} X_k\,(t, x, \varepsilon) \tag{2.2}$$

With the use of asymptotic methods it is possible to obtain the approximate solution

$$y = \bar{x} + \varepsilon U_1\,(t, \bar{x}) + \ldots + \varepsilon^k U_k\,(t, \bar{x}) \tag{2.3}$$

in which the principal term of expansion $\bar{x}$ satisfies the averaged equation

$$d\bar{x} / dt = \varepsilon X_0\,(\bar{x}) + \varepsilon^2 X_1\,(\bar{x}) + \ldots + \varepsilon^{k+1}\bar{X}_k\,(\bar{x}) = \varepsilon Z\,(\varepsilon, \bar{x}) \tag{2.4}$$

$$\left(\bar{X}_k(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T X_k(t, x, 0)\, dt\right) \tag{2.4}$$

We have to determine the quantity $\| y - \bar{y} \|$ For this we determine $\| \bar{x} - x \|$ between the solution of the exact equation (2.2) and the averaged equation (2.4).

Let us, first, assume that Eq. (2.4) has a quasi-static, i.e. independent of time, solution $\bar{x} = \xi$. The equation of perturbed motion for $\bar{x} = \xi$ can be written in the form

$$dh / dt = \varepsilon [Z(\xi + h, \varepsilon) - Z(\xi, \varepsilon)] = \varepsilon [A(\varepsilon)h + F(\varepsilon, h)] \tag{2.5}$$

$$A(\varepsilon) = \left.\frac{\partial Z(\bar{x}, \varepsilon)}{\partial \bar{x}}\right|_{\bar{x}=\xi}, \qquad \lim_{\|h\|\to 0} \frac{\| F(\varepsilon, h) \|}{\| h \|} = 0$$

Let among the eigenvalues $\lambda_q(A)$ of matrix $A(\varepsilon) = A_0 + \varepsilon A_1 + \ldots + \varepsilon^k A_k$ there be at least one lying in the right-hand half-plane, and

$$0 < \max \operatorname{Re} \lambda_q(A) < \nu, \quad \nu \leqslant \varepsilon^m a_m + \ldots + \varepsilon^k a_k \quad 0 \leqslant m \leqslant k \tag{2.6}$$

It is then possible to use the estimate [6]

$$\| e^{At} \| \leqslant N e^{\nu t} \tag{2.7}$$

We introduce in the analysis the quantity $u = (x - \xi) / \varepsilon^k$ substitute the new variable $\tau = \varepsilon t$ for t, and write the equation for u as

$$\frac{du}{d\tau} = \frac{1}{\varepsilon^k}[Z(\xi + \varepsilon^k u, \varepsilon) - Z(\xi, \varepsilon)] + \left[X_k\left(\frac{\tau}{\varepsilon}, \xi + \varepsilon^k u, \varepsilon\right) - \bar{X}_k(\xi + \varepsilon^k u)\right] \tag{2.8}$$

We rewrite (2.8) after separating the linear with respect to u part

$$du / d\tau = A(\varepsilon)u + 1/\varepsilon^k F(\varepsilon^k u, \varepsilon) + V(\tau/\varepsilon, u, \varepsilon) \tag{2.9}$$

$$V(\tau / \varepsilon, u, \varepsilon) = X_k(\tau / \varepsilon, \xi + \varepsilon^k u, \varepsilon) - \bar{X}_k(\xi + \varepsilon^k u)$$

where $A(\varepsilon)$ and $F(\varepsilon^k u, \varepsilon)$ are quantities defined in (2.5).

Theorem 1. Let Eq. (2.4) have a quasi-static solution that satisfies conditions (2.5) - (2.7), and let, furthermore, $X_k(\tau / \varepsilon, \varepsilon, x)$ converge as a whole to $\bar{X}_k(x)$, i.e. [6]

$$\lim_{\varepsilon\to 0} \int_{\tau_0}^{\tau_0+\tau} \left[X_k\left(\frac{\sigma}{\varepsilon}, \varepsilon, x\right) - \bar{X}_k(x)\right] d\sigma \tag{2.10}$$

Then

$$\lim_{\varepsilon\to 0} \sup_{x \in M(\varepsilon, T/\varepsilon^{m+1})} \max_{0 \leqslant t \leqslant T/\varepsilon^{m+1}} (\varepsilon^{-k} \| x(t) - \xi \|) = 0 \tag{2.11}$$

$$\xi \in M_k(\varepsilon, T/\varepsilon^{m+1})$$

Proof. We rewrite (2.9) in the form of the integral equation $u(\tau) = I_1(u, \tau, \varepsilon) + I_2(u, \tau, \varepsilon)$ where

$$I_1(u, \tau, \varepsilon) = \varepsilon^{-k} \int_0^\tau e^{A(\varepsilon)(\tau-\sigma)} F(\varepsilon^k u, \varepsilon)\, d\sigma$$

$$I_2(u, \tau, \varepsilon) = \int_0^\tau e^{A(\varepsilon)(\tau-\sigma)} V\left(\frac{\sigma}{\varepsilon}, u, \varepsilon\right) d\sigma$$

and estimate these integrals.

Because $F(\varepsilon, h)$ contains h in a higher power than the first, it is possible to choose the neighborhood $\| u \| < \rho$, such that

$$\varepsilon^{-k} \| F(\varepsilon^{k} u, \varepsilon) \| \leqslant q \| u \|, \quad q \leqslant \nu / N$$

We can then write

$$I_1(u, \tau, \varepsilon) \leqslant qN \int_0^\tau e^{\nu(\tau-\sigma)} \| u(\sigma) \| d\sigma \leqslant \nu \int_0^\tau e^{\nu(\tau-\sigma)} \| u(\sigma) \| d\sigma \tag{2.12}$$

To estimate the second term we integrate by parts, and obtain

$$I_2(u, \tau, \varepsilon) = \int_0^\tau e^{A(\varepsilon)(\tau-\sigma)} d_\sigma J(u, \sigma, \varepsilon) = J(u, \tau, \varepsilon) + \tag{2.13}$$

$$A(\varepsilon) \int_0^\tau e^{A(\varepsilon)(\tau-\sigma)} J(u, \sigma, \varepsilon) d\sigma, \quad J(u, \tau, \varepsilon) = \int_0^\tau V\left(\frac{s}{\varepsilon}, u, \varepsilon\right) ds$$

Because of condition (2.10) $\lim J(u, \sigma, \varepsilon) = 0$ when $\varepsilon \to 0$. However, according to theorem on limited convergence the transition to limit in (2.13) is only possible in the region where the quantity $\| e^{A(\varepsilon)(\tau-\sigma)} \|$, is limited, i.e. when $0 \leqslant \tau \leqslant T / \nu$ (see [7]. Then

$$I_2(u, \tau, \varepsilon) \leqslant \eta(\varepsilon) \tag{2.14}$$

and $\lim \eta(\varepsilon) = 0$ when $\varepsilon \to 0$ uniformly with respect to $u \in D$ and $\tau \in [0, T / \nu]$. Using (2.12) and (2.14) we obtain the integral inequality

$$\| u(\tau) \| \leqslant \nu \int_0^\tau e^{\nu(\tau-\sigma)} \| u(\sigma) \| d\sigma + \eta(\varepsilon)$$

from which we have [6]

$$\| u(\tau) \| \leqslant {}^1/_2 \eta(\varepsilon) (1 + e^{2\nu\tau}), \quad \| u(\tau) \| \to 0 \quad \text{при } 0 \leqslant \tau \leqslant T/\nu$$

where ν is of the form (2.6). The validity of formula (2.11) follows immediately from this.

For simplicity it was assumed that ξ is a quasi-static solution of Eq. (2.4). However the proof remains valid for any solution $\bar{x} = \bar{x}(t, \varepsilon)$ for which the Cauchy matrix of the variational equation

$$dh / dt = \varepsilon A(t, \varepsilon) h \tag{2.15}$$

satisfies the relationship

$$\| H(t, s) \| \leqslant N e^{\varepsilon \nu (t-s)} \tag{2.16}$$

and the expansion of the index ν begins from quantities of order ε^m (see (2.6). It is thus possible to assert the validity of the following theorem

Theorem 2. If Eq. (2.4) has the solution $\bar{x} = \bar{x}(t, \varepsilon)$ which satisfies conditions (2.15) and (2.16), then

$$\lim_{\varepsilon \to 0} \sup_{x(t) \in M(\varepsilon, T/\varepsilon^{m+1})} \max_{0 \leqslant t \leqslant T/\varepsilon^{m+1}} (\varepsilon^{-k} \| x(t) - \bar{x}(t) \|) = 0 \tag{2.17}$$

$$\bar{x}(t) \in M_k\left(\varepsilon, \frac{T}{\varepsilon^{m+1}}\right)$$

Let us estimate in conformity with [3] the quantity $\| y - \bar{y} \|$, where y is the exact solution of Eq. (1.1) defined by formula (1.11) and $\bar{y}$ is its asymptotic approximation of the form (2.3).

Formulas (1.16) imply that when conditions (P_k) are satisfied, the relationships $\| U_i (t, x) \|$, $\| \partial U_i (t, x) / \partial t \| \leqslant \psi (t)$, in which function $\psi (t)$ is bounded in every finite interval and $\lim t^{-1}\psi (t)=0$ when $t \to \infty$, are valid.

Moreover the statement

$$\| \varepsilon U_1 (t, x) + \ldots + \varepsilon^k U_k (t, x) \| \leqslant k\varepsilon\psi (t) \quad (0 \leqslant t < \infty)$$

is valid [3], i.e.

$$\| \varepsilon U_1 (t, x) + \ldots + \varepsilon^k U_k (t, x) \| \leqslant c (\varepsilon) \quad (0 \leqslant t \leqslant T/\varepsilon^{m+1}) \qquad (2.18)$$

and in the same way

$$\| \varepsilon \frac{\partial U_1}{\partial x} + \ldots + \varepsilon^k \frac{\partial U_k}{\partial x} \| \leqslant c (\varepsilon) \quad (0 \leqslant t \leqslant T/\varepsilon^{m+1}) \qquad (2.19)$$

and $\lim c (\varepsilon) = 0$ when $\varepsilon \to 0$.

Region D is called regular [3], if there exists a constant c such that any two points $x, y \in D$ can be connected by a straightened curve which is shorter than $c \| x - y \|$. Thus, when conditions (P_k) are satisfied and region D is regular, from (2.18) and (2.19) and from the definition of a regular region, follows that

$$\| y - \bar{y} \| = \| (x - \bar{x}) + [\varepsilon U_1 (t, x) + \ldots + \varepsilon^k U_k (t, x) - \varepsilon U_1 (t, \bar{x}) - \ldots - \varepsilon^k U_k (t, \bar{x})] \| \leqslant \| x - \bar{x} \| (1 + d (\varepsilon))$$

$$\left(\lim d (\varepsilon) = 0 \ \text{при} \ \varepsilon \to 0, \ x, \bar{x} \in D, \ 0 \leqslant t \leqslant \frac{T}{\varepsilon^{m+1}}\right)$$

Since the quantity $\| \bar{x} - x \|$ satisfies condition (2.17), we can write

$$\lim_{\varepsilon \to 0} \sup_{y(t) \in N(\varepsilon, T/\varepsilon^{m+1})} \max_{0 \leqslant t \leqslant T/\varepsilon^{m+1}} (\varepsilon^{-k} \| y (t) - \bar{y} (t) \|) = 0$$

$$\bar{y} (t) \in N_k \left(\varepsilon, \frac{T}{\varepsilon^{m+1}}\right)$$

where $N (\varepsilon, T)$ is the set of all solutions of Eq. (1.1) determined in $[0, T]$ which satisfy the initial condition $y (0) = y_0$, and $N_k (\varepsilon, T)$ is the set of all asymptotic approximations of the $k + 1$- st approximation of solution $y (t)$.

It can be shown that the proof of the averaging method for infinite time interval [3] is also based on Theorems 1 and 2. For this it is sufficient to repeat the proof by setting $\nu < 0$ in (2.6) and (2.16).

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